Given an elastic body containing point P

A load is applied and the part deflects. Each material point P will move to a new position P' unless constrained by supports.

Take a closer look at point P. P moves from point \((x, y)\) to point \((x', y')\) along vector \(\vec{u}\) where

\[
x' = x + u_x = x + u
\]

\[
y' = y + u_y = y + v
\]

\(\vec{u} = \text{vector motion from point } P \text{ to point } P'\)

\(\vec{u}^* = \text{displacement vector}\)
ME 3401

2D: \( \mathbf{u} = u_x \mathbf{i} + u_y \mathbf{j} = u^x + v^y \)

3D: \( \mathbf{u} = u_x \mathbf{i} + u_y \mathbf{j} + u_z \mathbf{k} = u^x + v^y + w^z \)

Note: The displacement vector \( \mathbf{u} \) will vary from point to point in the structure.

Note: Displacement refers to all components \( u, v, w \). Deflection is used to denote a special displacement component \( w \) at a specific point on a structure.

Complete information of displacement for every point in a body requires that \( u, v, \) and \( w \) be given as equations which are functions of material coordinates \( x, y, \) and \( z \)

\[
\begin{align*}
    u &= u(x, y, z) \\
    v &= v(x, y, z) \\
    w &= w(x, y, z)
\end{align*}
\]

Since the displacements are not the same from point to point within a body under load, the body will stretch and change shape; it will deform. Deformations of a body are called strains and, like displacement, strain varies from point to point.
Strain is nondimensional. Its units are displacement per unit length \((\text{mm/mm or in/in})\).

*Example - Extension of a bar (normal strain):*

\[ e = \frac{\Delta L}{L} \]

*For small displacements \(\Delta L\):*

*Example - Shear loading (shear strain):*

Shear strain measures the change in angle \(\gamma\) (measured in radians).

*For small \(\Delta\):*

\[ \tan \gamma \approx \gamma = \frac{\Delta}{D} \]
Positive shear results in a reduction of angle from 90°.

Before

After (deformed shape, positive shear)

Given a small, originally rectangular element of side lengths $\Delta x$ and $\Delta y$.

Load the element and the 4 corners will be displaced by different amounts.
Displacement = Displacement of corner 1 plus the rate of change in a coordinate direction times the element's edge length in that direction.

Normal strain along an axis is given by
\[
\varepsilon_x = \frac{\Delta L}{L} = \frac{u + \frac{\partial u}{\partial x} \Delta x - u}{\Delta x} = \frac{\partial u}{\partial x} = \varepsilon_x
\]

for \( x \) or \( y \) (corners 2 or 4 - corner 1)
\[
\varepsilon_y = \frac{\Delta L}{L} = v + \frac{\partial v}{\partial y} \Delta y - v = \frac{\partial v}{\partial y} = \varepsilon_y
\]

Engineering shear strain, \( \gamma_{xy} \)
\[
\gamma_{xy} = \text{total angle decrease in corner 1}
\]
Along the x-axis (corners 1 to 2), the rotation is
\[
v + \frac{\partial v}{\partial x} \Delta x - v = \frac{\partial v}{\partial x}
\]
And along the y-axis (corners 1 to 4), the rotation is
\[
u + \frac{\partial u}{\partial y} \Delta y - u = \frac{\partial u}{\partial y}
\]
Thus, the total rotation $\delta y$ is the sum of those

$$\delta y_x = \frac{\partial \delta y}{\partial y} + \frac{\partial \delta y}{\partial x}.$$ 

So in 2D, strain is given by

$$\varepsilon_x = \frac{\partial u}{\partial x} \quad (1)$$
$$\varepsilon_y = \frac{\partial v}{\partial y} \quad (2)$$
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad (3)$$

Thus:

1. If the displacement field $u(x, y)$ and $v(x, y)$ is known throughout a body, the strain field can be calculated throughout the body.

2. Similarly, if the strain field is known throughout the body as a function of $x$ and $y$, then equations (1) $\rightarrow$ (3) can be integrated and, with appropriate boundary conditions and consideration of compatibility, the displacement field may be found.
Strain in 3D

Displacement vector now has three components:

\[ \vec{u} = u_x \hat{i} + u_y \hat{j} + u_z \hat{k} \]
\[ \vec{u} = u_x \hat{i} + v_y \hat{j} + w_z \hat{k} \]

\[ u = x{-}\text{displacement} \]
\[ v = y{-}\text{displacement} \]
\[ w = z{-}\text{displacement} \]

3 Normal Strains (3 directions for elongation):

\[ \varepsilon_x \quad \varepsilon_y \quad \varepsilon_z \]

3 Shear Strains (3 angle changes):

\[ \gamma_{xy} \quad \gamma_{yz} \quad \gamma_{xz} \]

Normal Strains are found by calculating \( \frac{1}{2} \)
along an axis, giving:

\[ \varepsilon_x = \frac{\partial u_x}{\partial x} \]
\[ \varepsilon_y = \frac{\partial v_y}{\partial y} \]
\[ \varepsilon_z = \frac{\partial w_z}{\partial z} \]

Shear Strains are found by calculating the
total change in angle, giving:

\[ \gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial v_y}{\partial x} \]
\[ \gamma_{y2} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial y} \quad (5) \]

\[ \gamma_{x2} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \quad (6) \]

**Tensoral Strain**

The transformation equations of strain do not have the same form as those for stress when engineering shear strain, \( \gamma \), is used. Therefore, we introduce the tensor for \( \gamma \) shear strain

\[ \varepsilon_{xy} = \frac{1}{2} \gamma_{xy} = \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial w}{\partial x} \right) \]

\[ \varepsilon_{yx} = \frac{1}{2} \gamma_{yx} = \frac{1}{2} \left( \frac{\partial u}{\partial x} + \frac{\partial w}{\partial y} \right) \]

\[ \varepsilon_{xz} = \frac{1}{2} \gamma_{xz} = \frac{1}{2} \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \]

If we use the tensoral strains, \( \varepsilon_{xy}, \varepsilon_{xz}, \varepsilon_{yz} \), then we can use the same transformation equations for strain as those we derived for stress.

Strain tensor \([\varepsilon] = [\varepsilon_{ij}] = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & \frac{1}{2} \varepsilon_{y} & \frac{1}{2} \varepsilon_{z} \\ \frac{1}{2} \varepsilon_{y} & \varepsilon_{yy} & \frac{1}{2} \varepsilon_{y} \\ \frac{1}{2} \varepsilon_{z} & \frac{1}{2} \varepsilon_{y} & \varepsilon_{zz} \end{bmatrix} \]
Engineering Strains in matrix form

\[
\begin{bmatrix}
\varepsilon_x & \gamma_{xy} & \gamma_{xz} \\
\gamma_{yx} & \varepsilon_y & \gamma_{yz} \\
\gamma_{zx} & \gamma_{zy} & \varepsilon_z \\
\end{bmatrix}
\]

\[
\begin{align*}
\varepsilon_x &= \frac{\partial u}{\partial x} \\
\varepsilon_y &= \frac{\partial v}{\partial y} \\
\varepsilon_z &= \frac{\partial w}{\partial z} \\
\gamma_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \\
\gamma_{yz} &= \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \\
\gamma_{zx} &= \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \\
\end{align*}
\]

Tensorial Strain in matrix form

\[
\begin{bmatrix}
\varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\
\varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\
\varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \\
\end{bmatrix}
\]

\[
\begin{align*}
\varepsilon_{xx} &= \frac{1}{2} \gamma_{xy} = \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \\
\varepsilon_{yy} &= \frac{1}{2} \gamma_{xx} = \frac{1}{2} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \\
\varepsilon_{zz} &= \frac{1}{2} \gamma_{yy} = \frac{1}{2} \left( \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \\
\varepsilon_{xy} &= \frac{1}{2} \gamma_{yx} = \frac{1}{2} \left( \frac{\partial v}{\partial y} + \frac{\partial w}{\partial y} \right) \\
\end{align*}
\]

The equations derived for stress transformation can be applied to strain transformation provided the tensorial strain is used.

In 2D:

\[
\begin{align*}
\varepsilon_{xx}' &= \varepsilon_{xx} \cos^2 \theta + \varepsilon_{yy} \sin^2 \theta + 2 \varepsilon_{xy} \sin \theta \cos \theta \\
\varepsilon_{yy}' &= \varepsilon_{xx} \sin^2 \theta + \varepsilon_{yy} \cos^2 \theta - 2 \varepsilon_{xy} \sin \theta \cos \theta \\
\varepsilon_{xy}' &= (\varepsilon_{yy} - \varepsilon_{xx}) \sin \theta \cos \theta + \varepsilon_{xy} (\cos^2 \theta - \sin^2 \theta) \\
\end{align*}
\]

\text{CW rotation } \Rightarrow \text{ Positive } \theta \\
\text{CW rotation } \Rightarrow \theta \text{ is negative}
In 3D,

\[
[e'] = [\eta][e][\eta]^T
\]

where, as before,

\[
[\eta] = \begin{bmatrix}
\eta_{xx} & \eta_{xy} & \eta_{xz} \\
\eta_{yx} & \eta_{yy} & \eta_{yz} \\
\eta_{zx} & \eta_{zy} & \eta_{zz}
\end{bmatrix}
\]

direction cosine matrix

and

\[
[e] = \begin{bmatrix}
E_{xx} & E_{xy} & E_{xz} \\
E_{yx} & E_{yy} & E_{yz} \\
E_{zx} & E_{zy} & E_{zz}
\end{bmatrix}
\]

tensorial strains

or

\[
e_{xx}' = E_{xx} \eta_{xx}^2 + E_{yy} \eta_{yy}^2 + E_{zz} \eta_{zz}^2
\]

\[
+ 2(E_{xy} \eta_{xx} \eta_{xy} + 2E_{xz} \eta_{xx} \eta_{xz} + 2E_{yz} \eta_{yy} \eta_{yz})
\]

\[
+ 2(E_{xz} \eta_{xx} \eta_{xz})
\]

e etc.

(See notes on 3D stress transformations. Strain transformations are obtained by substituting

\[
\sigma_{xx} \text{ with } E_{xx}
\]

\[
\sigma_{yy} \text{ with } E_{yy}
\]

\[
\tau_{zz} \text{ with } E_{zz}
\]

\[
\tau_{xy} \text{ with } E_{xy}
\]

\[
\tau_{xz} \text{ with } E_{xz}
\]

and

\[
\tau_{yz} \text{ with } E_{yz}
\]

\[
\tau_{zxy} \text{ with } E_{zxy}
\]
Strain Transformation Example

Strain gage Rosettes  0-45-90 Rosette

Each gage measures strain in one direction only.

Angle the gage makes with the x-axis

Gage A:  \( \theta_A = \)

Gage B:  \( \theta_B = \)

Gage C:  \( \theta_C = \)

\( \varepsilon_{gage} = \varepsilon_{xx} \cos^2 \theta + \varepsilon_{yy} \sin^2 \theta + 2 \varepsilon_{xy} \sin \theta \cos \theta \)

Gage A:  \( \varepsilon_A = \)

Gage C:  \( \varepsilon_C = \)

Gage B:  \( \varepsilon_B = \)
Plane Strain strain acts in one plane only

Example

Long transversely loaded pipe
No strain develops in the z-direction

\[ \varepsilon_{xz} = \varepsilon_{yz} = \varepsilon_{zz} = 0 \]

State of plane strain

Principal Strains

Principal Strains are the roots of the equation

\[ \varepsilon_1^3 - I_{1e} \varepsilon_1^2 + I_{2e} \varepsilon_1 - I_{3e} = 0 \]

\[ \varepsilon_1 > \varepsilon_2 > \varepsilon_3 \]

where

\[ I_{1e} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} \]
\[ I_{2e} = \varepsilon_{xx} \varepsilon_{yy} + \varepsilon_{xx} \varepsilon_{zz} + \varepsilon_{yy} \varepsilon_{zz} - \varepsilon_{xy}^2 - \varepsilon_{yz}^2 - \varepsilon_{xz}^2 \]
\[ = \varepsilon_{xx} \varepsilon_{yy} + \varepsilon_{xx} \varepsilon_{zz} + \varepsilon_{yy} \varepsilon_{zz} - \frac{\varepsilon_{xy}^2}{4} - \frac{\varepsilon_{yz}^2}{4} - \frac{\varepsilon_{xz}^2}{4} \]
\[ I_{3e} = \varepsilon_{xx} \varepsilon_{yy} \varepsilon_{zz} + 2 \varepsilon_{xy} \varepsilon_{yz} \varepsilon_{xz} - \varepsilon_{xx} \varepsilon_{yz}^2 - \varepsilon_{yy} \varepsilon_{xz}^2 \]
\[ - \varepsilon_{zz} \varepsilon_{xy}^2 \]

\[ I_{1e}, I_{2e}, I_{3e} \] are the strain invariants. Their values do not vary with coordinate system.
Likewise, as for principal stresses, we can solve for the direction cosines
\[ \begin{align*}
\eta_{\text{max}}, \eta_{\text{y}}, \eta_{\text{z}} & \text{ for } \epsilon_1 \\
\eta_{\text{z}}, \eta_{\text{y}}, \eta_{\text{x}} & \text{ for } \epsilon_2 \\
\eta_{\text{x}}, \eta_{\text{z}}, \eta_{\text{y}} & \text{ for } \epsilon_3
\end{align*} \]

Max tensor shear strain
\[ \epsilon_{\text{max}} = \frac{\epsilon_1 - \epsilon_3}{2} \]

Max engineering shear strain
\[ \gamma_{\text{max}} = 2 \epsilon_{\text{max}} = \epsilon_1 - \epsilon_3 \]

Stress-Strain Relations

Hooke's law for linear elastic, isotropic, homogeneous materials is
\[ \sigma_x = E \epsilon_x \]

Where \( E \) = Modulus of Elasticity (Young's Modulus) \( E \) has units of pressure.
As the bar elongates in \( x \), it contracts in \( y+z \).

Poisson's ratio relates the contraction in \( y+z \) to the elongation in \( x \):

\[
Poisson's\ Ratio = \nu = \frac{\text{transverse strain}}{\text{longitudinal strain}}
\]

\[
\varepsilon_y = -\nu \varepsilon_x = -\nu \frac{\sigma_x}{E}
\]
\[
\varepsilon_z = -\nu \varepsilon_x = -\nu \frac{\sigma_x}{E}
\]

Similarly, if the bar was instead loaded in the \( y \)-direction:

\[
\varepsilon_y = \frac{\sigma_y}{E}
\]
\[
\varepsilon_x = \varepsilon_z = -\nu \varepsilon_y = -\nu \frac{\sigma_y}{E}
\]

If an element undergoes \( \sigma_x, \sigma_y \) and \( \sigma_z \) simultaneously, using superposition, we get:
$e_x = \frac{1}{E} \left[ \sigma_x - \nu \sigma_y - \nu \sigma_z \right]$

$e_y = \frac{1}{E} \left[ \sigma_y - \nu \sigma_x - \nu \sigma_z \right]$

$e_z = \frac{1}{E} \left[ \sigma_z - \nu \sigma_x - \nu \sigma_y \right]$

Or, solving for stress...

$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)} \left[ (1-\nu) e_x + \nu (e_y + e_z) \right]$

$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)} \left[ (1-\nu) e_y + \nu (e_x + e_z) \right]$

$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)} \left[ (1-\nu) e_z + \nu (e_x + e_y) \right]$

**Shear Stress and Strain**

For linear elastic, homogeneous, isotropic materials

$\gamma_{xy} = \frac{1}{G} \epsilon_{xy}$

$\gamma_{xy}$ measured in radians

$G =$ Shear Modulus (units of pressure)

$G = \frac{E}{2(1+\nu)}$
Likewise
\[ \varepsilon_{xx} = \frac{\tau_{xx}}{G} \quad \varepsilon_{yy} = \frac{\tau_{yy}}{G} \]

**Stress - Strain Equations for plane stress**

\[ \sigma_x = \tau_{xx} = \tau_{yy} = 0 \]

\[ \varepsilon_x = \frac{1}{E} \left[ \sigma_x - v\sigma_y \right] \]
\[ \varepsilon_y = \frac{1}{E} \left[ \sigma_y - v\sigma_x \right] \]
\[ \varepsilon_z = \frac{1}{E} \left[ -\sigma_x - \sigma_y \right] \]

Note: \( \varepsilon_z \neq 0 \) for plane stress case!

\[ \gamma_{xy} = \frac{\tau_{xy}}{G} \]
\[ \gamma_{yz} = \gamma_{zx} = 0 \]

Substituting \( \varepsilon_z = -\frac{1}{E} (\sigma_x + \sigma_y) \) into the 3D stress-strain equations gives, for **plane stress**

\[ \sigma_x = \frac{E}{1-v^2} (\varepsilon_x + v\varepsilon_y) \]
\[ \sigma_y = \frac{E}{1-v^2} (\varepsilon_y + v\varepsilon_x) \]
\[ \tau_{xy} = G \gamma_{xy} \]

\[ \{ \text{Plane Stress} \quad \text{Only}! \} \]
Stress-Strain Equations for Plane Strain

\[ \varepsilon_x = \gamma_{x\varepsilon} = \gamma_{yx} = 0 \]

\[ \sigma_x = \frac{E}{(1+\nu)(1-2\nu)} \left[ (1-\nu)\varepsilon_x + \nu \varepsilon_y \right] \]

\[ \tau_y = \frac{E}{(1+\nu)(1-2\nu)} \left[ (1-\nu)\varepsilon_y + \nu \varepsilon_x \right] \]

\[ \sigma_z = \frac{E\nu}{(1+\nu)(1-2\nu)} (\varepsilon_x + \varepsilon_y) \quad \text{Note:} \sigma_z \neq 0 \text{ for plane strain case!} \]

\[ T_{xy} = G \gamma_{xy} \]

\[ T_{xz} = T_{yz} = 0 \]