Appendix to Economies of Scale in Banking, Confidence Shocks, and Business Cycles

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Abstract

This appendix details: (i) a version of the model with general preferences and internal ES in the intermediary; (ii) the model solution via Lubik and Schorfheide (2003); and (iii) the calibration exercise.

1. The Model and Equilibrium

1.1. The Model

Time is discrete and the horizon is infinite. The economy is populated by a continuum of households indexed by $i \in [0, 1]$ which supply differentiated labor, a continuum of industries indexed by $j \in [0, 1]$ which produce differentiated goods and have a large number of perfectly-competitive firms within each industry, a financial intermediary, and a monetary authority. The model contains enough symmetry to allow the analysis to focus on a representative household i, and a single firm in representative industry j.

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Households

The preferences of household i are given by

$$E_0 \sum_{t=0}^{\infty} \beta^t u\left(c^i(t), h^i(t)\right), \tag{1}$$

where $c^i(t)$ is a composite consumption good, $h^i(t) = \int_0^1 h_j^i(t)\, dj$ is labor supply across industries, and $\beta \in (0,1)$ is the discount rate. Composite consumption is an (Armington) aggregate of differentiated goods given by the CES specification

$$c^i(t) = \left[\int_0^1 \varphi_j^{\frac{1}{\varpi}} c_j^i(t)^{\frac{\varpi-1}{\varpi}}\, dj\right]^{\frac{\varpi}{\varpi-1}}, \tag{2}$$

where $c_j^i(t)$ denotes household i's consumption of good j and $\varpi \geq 0$ denotes the constant elasticity of substitution across goods. The parameter φ_j denotes the weight associated with good j in the consumption set, and must satisfy $\int_0^1 \varphi_j dj = 1$.

Household i begins period t with physical capital $k^i(t)$ and nominal currency $M^i(t)$. Every household receives a lump-sum transfer $T(t)$ of currency from the monetary authority, and buys / sells nominal bonds $B^i(t)$ which are zero in net supply and earn a gross nominal return $1+R(t)$. The household then deposits $d^i(t)$ of its capital into a financial intermediary earning a gross real return $r_d(t)$, and lends $a^i(t)$ directly to firms earning a gross real return $r(t)$. Therefore, $k^i(t) = a^i(t) + d^i(t)$.

Both deposits and currency can be used to purchase consumption. As in the standard cash-in-advance model, previously held currency can costlessly purchase consumption goods. Deposits are chosen at the beginning of the period and pay interest, but bear a fixed real cost γ for each consumption good purchased. This cost can be interpreted as a per-check processing cost.

The use of money balances deliver the conditions

$$M^i(t) + T(t) - B^i(t) \geq \int_0^1 \mathbf{1}_{Jm}(j)\, P_j(t)\, c_j^i(t)\, dj, \tag{3}$$

$$P_k(t)\, d^i(t) \geq \int_0^1 \mathbf{1}_{Jd}(j)\, P_j(t)\, c_j^i(t)\, dj, \tag{4}$$

where $P_j(t)$ denotes the price of consumption good j, $P_k(t)$ is the price of capital (and capital deposits), and Jm and Jd are subsets of $[0, 1]$ which denote the good types purchased with currency and deposits, respectively. The indicator function $\mathbf{1}_{Jm}(j)$ ($\mathbf{1}_{Jd}(j)$) equals one if a particular good j is a member of Jm (Jd), and zero otherwise.

Household i is a monopoly supplier of type-i labor which is sold to all firms. Different types of labor are imperfect substitutes in production, so labor is sold in a monopolistically-competitive market: household i sets the nominal wage $W_j^i(t)$ offered to a representative firm from industry j (henceforth, firm j) and supplies labor such that it satisfies firm j's demand taking all prices as given. It is assumed that the household faces a quadratic cost to adjust its nominal wage as in Rotemberg (1982),

$$\frac{\phi}{2}\left[\frac{W_j^i(t)}{\pi W_j^i(t-1)} - 1\right]^2,$$

where $\phi > 0$ governs the size of the real adjustment cost and π denotes the gross, long-run inflation rate.

The flow budget constraint of household i is given by

$$\begin{gathered} \int_0^1 P_j(t)\, c_j^i(t)\, dj + M^i(t+1) + P_k(t)\, k^i(t+1) \leq \\ \int_0^1 W_j^i(t)\, h_j^i(t)\, dj + r(t)\, P_k(t)\, a^i(t) + r_d(t)\, P_k(t)\, d^i(t) + R(t)\, B^i(t) + M^i(t) + T(t) \\ -P_k(t)\, \gamma\left(\int_0^1 \mathbf{1}_{Jd}(j)\, dj\right) - P_k(t) \int_0^1 \frac{\phi}{2}\left[\frac{W_j^i(t)}{\pi W_j^i(t-1)} - 1\right]^2 dj \end{gathered} \tag{5}$$

where $\gamma\left(\int_0^1 \mathbf{1}_{Jd}(j)\, dj\right)$ denotes the total cost of using deposits.

Production

A representative type-j firm hires differentiated labor from households and aggregates them into a homogeneous labor input $h_j(t)$ using the CES technology:

$$h_j(t) = \left(\int_0^1 h_j^i(t)^{\frac{\xi-1}{\xi}} di \right)^{\frac{\xi}{\xi-1}}, \tag{6}$$

where $\xi \geq 0$ denotes the elasticity of substitution between labor types.[1]

The production technology for type-j output is a CRS function of capital and homogeneous labor: $y_j(t) = f(z(t), k_j(t), h_j(t))$, where $z(t)$ denotes exogenous total factor productivity which is identical across firms and evolves according to $z(t) = \kappa_z + \rho_z z(t-1) + \varepsilon_z(t)$ with $\varepsilon_z(t) \sim N(0, \sigma_z^2)$. Profits of a representative type-j firm are given by

$$P_j(t) y_j(t) + (1 - \delta - r(t)) P_k(t) k_j(t) - \int_0^1 W_j^i(t) h_j^i(t) di, \tag{7}$$

where $P_j(t)$ is taken as given.

Financial Intermediaries

The financial intermediary accepts capital deposits from households and loans them to firms. No financial frictions are assumed. As discussed in the text, a benefit of holding deposits rather than currency is the earned interest r_d, while a benefit of deposits relative to direct capital investment is the ability to purchase consumption (for a processing cost γ).

The profit function of an intermediary is given by

$$r(t) d(t) - r_d(t) d(t) - C\left(d(t), \overline{d}(t)\right), \tag{8}$$

where $d(t)$ denotes real deposits, $\overline{d}(t)$ denotes real deposits of the entire intermediary sector,

[1] One could establish an equivalent environment where an additional production sector aggregates labor and sells homogeneous labor units to good producing firms as in Erceg et al. (2000). Allowing firms to hire heterogeneous labor is employed here simply to streamline the environment.

and $C\left(d\left(t\right),\overline{d}(t)\right)$ denotes real operating costs. Let $C\left(d\left(t\right),\bar{d}(t)\right) = \Gamma d\left(t\right)\bar{d}(t)^{\theta}$. The intermediary takes $r(t)$, $r_d(t)$ and $\bar{d}(t)$ as given and chooses $d(t)$ to equate marginal costs with benefits.

$$r_d\left(t\right) = r\left(t\right) - \Gamma\bar{d}(t)^{\theta} \tag{9}$$

The cost function of the intermediary exhibits ES for $\theta < 0$ and (9) suggests that the rate of returns on deposits is an increasing function of the aggregate amount of deposits (all else constant).

1.2. Household i's Generalized and Aggregated Problems

The generalized problem of household i can be stated as

$$\begin{aligned}
&\max\sum_{t=1}^{\infty}\beta^{t}\{u\left[c^{i}\left(t\right),h^{i}\left(t\right)\right] \\
&+\lambda_1^i\left(t\right)\left[M^i\left(t\right)+T\left(t\right)-\textstyle\int_0^1\mathbf{1}_{Jm}\left(j\right)P_j\left(t\right)c_j^i\left(t\right)dj\right] \\
&+\lambda_2^i\left(t\right)\left[P_k\left(t\right)d^i\left(t\right)-\textstyle\int_0^1\mathbf{1}_{Jd}\left(j\right)P_j\left(t\right)c_j^i\left(t\right)dj\right] \\
&+\lambda_3^i\left(t\right)\left[\begin{array}{c}
\int_0^1 W_j^i\left(t\right)h_j^i\left(t\right)dj+r\left(t\right)P_k\left(t\right)\left[k^i(t)-d^i(t)\right]+r_d\left(t\right)P_k\left(t\right)d^i\left(t\right) \\
+M^i\left(t\right)+T\left(t\right)+R\left(t\right)B^i\left(t\right)-P_k\left(t\right)\gamma\left(\int_0^1\mathbf{1}_{Jd}\left(j\right)dj\right)-P_k\left(t\right)\int_0^1\frac{\phi}{2}\left[\frac{W_j^i(t)}{\pi W_j^i(t-1)}-1\right]^2dj \\
-\int_0^1 P_j\left(t\right)c_j^i\left(t\right)dj-M^i\left(t+1\right)-P_k\left(t\right)k^i\left(t+1\right)
\end{array}\right]\Bigg\},
\end{aligned}$$

where $c^i(t) = \left[\int \varphi_j^{\frac{1}{\varpi}} c_j^i(t)^{\frac{\varpi-1}{\varpi}} dj\right]^{\frac{\varpi}{\varpi-1}}$. The first order conditions for choices of $c_j^i(t)\ \forall j \in Jm$, $c_{j'}^i(t)\ \forall j' \in Jd$, $d^i(t)$, $B^i(t)$, $M^i(t+1)$, $k^i(t+1)$ and $W_j^i(t)\ \forall j$ are given by

$$u_{c_j^i}(t)\left(c^i(t)\varphi_j\right)^{\frac{1}{\varpi}} = c_j^i(t)^{\frac{1}{\varpi}} P_j(t)\left[\lambda_1^i(t)+\lambda_3^i(t)\right],\ \forall j \in Jm, \tag{10}$$

$$u_{c_{j'}^i}(t)\left(c^i(t)\varphi_{j'}\right)^{\frac{1}{\varpi}} = c_{j'}^i(t)^{\frac{1}{\varpi}} P_{j'}(t)\left[\lambda_2^i(t)+\lambda_3^i(t)\right],\ \forall j' \in Jd, \tag{11}$$

$$\lambda_2^i(t) = \lambda_3^i(t)\left[r(t)-r_d(t)\right], \tag{12}$$

$$\lambda_1^i(t) = \lambda_3^i(t) R(t), \tag{13}$$

$$\lambda_3^i(t) = \beta E(t)\left[\lambda_1^i(t+1)+\lambda_3^i(t+1)\right], \tag{14}$$

$$\lambda_3^i(t) = \beta E(t) r(t+1)\lambda_3^i(t+1), \tag{15}$$

and

$$u_{h_j^i}(t)\xi h_j(t)\left(\frac{W_j(t)}{W_j^i(t)}\right)^{\xi} - \lambda_3^i(t)\begin{bmatrix}(1-\xi)h_j(t)W_j^i(t)\left(\frac{W_j(t)}{W_j^i(t)}\right)^{\xi}\\ -P_k(t)\phi\left[\frac{W_j^i(t)}{\pi W_j^i(t-1)}\right]\frac{W_j^i(t)}{\pi W_j^i(t-1)}\end{bmatrix}$$
$$= \beta E(t)\lambda_3^i(t+1)\left[P_k(t+1)\phi\left[\frac{W_j^i(t+1)}{\pi W_j^i(t)}\right]\frac{W_j^i(t+1)}{\pi W_j^i(t)}\right],\ \forall j. \tag{16}$$

The aggregated problem of household i can be stated as

$$\begin{aligned}
&\max \sum_{t=1}^{\infty}\beta^t\{u\left[c^i(t), h^i(t)\right]\\
&+\hat{\lambda}_1^i(t)\left[M^i(t)+T(t)-P(t)c^i(t)\textstyle\int_0^1 \mathbf{1}_{Jm}(j)\varphi_j dj\right]\\
&+\hat{\lambda}_2^i(t)\left[P_k(t)d^i(t)-P(t)c^i(t)\textstyle\int_0^1 \mathbf{1}_{Jd}(j)\varphi_j dj\right]\\
&+\hat{\lambda}_3^i(t)\left.\begin{bmatrix}\int_0^1 W_j^i(t)h_j^i(t)dj + r(t)P_k(t)\left[k^i(t)-d^i(t)\right]+r_d(t)P_k(t)d^i(t)\\ +M^i(t)+T(t)+R(t)B^i(t)-P_k(t)\gamma\left(\int_0^1 \mathbf{1}_{Jd}(j)dj\right)-P_k(t)\int_0^1\frac{\phi}{2}\left[\frac{W_j^i(t)}{\pi W_j^i(t-1)}-1\right]^2 dj\\ -P(t)c^i(t)-M^i(t+1)-P_k(t)k^i(t+1)\end{bmatrix}\right\},
\end{aligned}$$

and the first order condition for the choice of $c^i(t)$ is given by

$$u_{c^i}(t) = P(t)\left[\hat{\lambda}_3^i(t) + \hat{\lambda}_1^i(t)\int_0^1 \mathbf{1}_{Jm}(j)\,\varphi_j dj + \hat{\lambda}_2^i(t)\int_0^1 \mathbf{1}_{Jd}(j)\,\varphi_j dj\right]. \tag{17}$$

The remaining first order conditions (with the exception of the multipliers) are identical to the generalized problem.

Deriving the aggregate price and consumption demand equations begins with the claim (and verification) that the problems above are equivalent. This claim implies that the multipliers are equivalent (e.g. $\hat{\lambda}_3^i = \lambda_3^i$). Use (10) and (11) to solve for $\lambda_1^i(t)$ and $\lambda_2^i(t)$. This requires repeated use of (2) and integrating both sides with respect to j.

$$\lambda_1^i(t) = u_{c_j^i}(t)\left[\int_0^1 \varphi_j P_j(t)^{1-\varpi}\,dj\right]^{\frac{1}{\varpi-1}} - \lambda_3^i(t)\,,\ \forall j \in Jm \tag{18}$$

$$\lambda_2^i(t) = u_{c_{j'}^i}(t)\left[\int_0^1 \varphi_{j'} P_{j'}(t)^{1-\varpi}\,dj\right]^{\frac{1}{\varpi-1}} - \lambda_3^i(t)\,,\ \forall j' \in Jd \tag{19}$$

Substitution of these multipliers into (17) results in

$$u_{c^i}(t) = P(t)\left[\begin{array}{c} \lambda_3^i(t) + \left[u_{c_j^i}(t)\left[\int_0^1 \varphi_j P_j(t)^{1-\varpi}\,dj\right]^{\frac{1}{\varpi-1}} - \lambda_3^i(t)\right]\int_0^1 \mathbf{1}_{Jm}(j)\,\varphi_j dj \\ +\left[u_{c_{j'}^i}(t)\left[\int_0^1 \varphi_{j'} P_{j'}(t)^{1-\varpi}\,dj\right]^{\frac{1}{\varpi-1}} - \lambda_3^i(t)\right]\int_0^1 \mathbf{1}_{Jd}(j)\,\varphi_j dj \end{array}\right]. \tag{20}$$

Since Jm and Jd span the set of goods, $\lambda_3^i(t)$ and $u_{c^i}(t)$ drops out leaving

$$P(t) = \left[\int_0^1 \varphi_j P_j(t)^{1-\varpi}\,dj\right]^{\frac{1}{1-\varpi}}. \tag{21}$$

Verifying that the multipliers are equal (and the problems are equivalent) can be done by verifying that $P(t)\,c^i(t) = \int_0^1 P_j(t)\,c_j^i(t)\,dj$. Using only the generalized problem, replacing either $\lambda_1^i(t)$ in (10) with its expression in (18) or $\lambda_2^i(t)$ in (11) with its expression in (19) results in

$$\left(c^{i}\left(t\right)\varphi_{j}\right)^{\frac{1}{\varpi}}=c_{j}^{i}\left(t\right)^{\frac{1}{\varpi}}P_{j}\left(t\right)\left[\int_{0}^{1}\varphi_{j}P_{j}\left(t\right)^{1-\varpi}dj\right]^{\frac{1}{\varpi-1}}.$$

Raising both sides to the power ϖ, rearranging terms, integrating both sides with respect to j, and using (21) verifies the result and delivers

$$c_{j}^{i}\left(t\right)=\left(\frac{P\left(t\right)}{P_{j}\left(t\right)}\right)^{\varpi}\varphi_{j}c^{i}\left(t\right). \tag{22}$$

It is assumed for simplicity that the differentiated consumption goods are perfect complements (i.e. $\varpi\rightarrow0$) and the consumption weights are chosen to deliver an ordinal ranking of consumption types. Letting $\varphi_{j}=2j$, equations (21) and (22) become

$$P\left(t\right)=\int_{0}^{1}\left(2j\right)P_{j}\left(t\right)dj, \tag{23}$$

$$c_{j}^{i}\left(t\right)=\left(2j\right)c^{i}\left(t\right). \tag{24}$$

(24) is equation (3) in the text.

Therefore, the price index is a weighting of differentiated prices, and the demand for each good is its weighted contribution to total consumption. These simplifying assumptions on (2) reduce the general preferences considered here to those considered by Freeman and Kydland (2000) and Dressler (2007). Note the smaller the value of j, the smaller the contribution $c_{j}^{i}\left(t\right)$ is to $c^{i}\left(t\right)$.

Under the aggregated problem, household optimization is characterized by the binding constraint set and the Euler equations

$$u_{c^i}(t)\,\Psi^i(t) = \beta E(t)\, r(t+1)\, u_{c^i}(t+1)\,\Psi^i(t+1)\,, \tag{25}$$

$$u_{c^i}(t)\,\Psi^i(t) = \beta E(t)\,\frac{1+R(t+1)}{\pi(t+1)} u_{c^i}(t+1)\,\Psi^i(t+1)\,, \tag{26}$$

$$u_{h_j^i}(s_t)\,\xi h_j(s_t)\left(\frac{W_j(s_t)}{W_j^i(s_t)}\right)^{\xi} = u_{c^i}(t)\,\Psi^i(t)\left[\begin{array}{c}(1-\xi)\,h_j(s_t)\,\frac{W_j^i(s_t)}{P(s_t)}\left(\frac{W_j(s_t)}{W_j^i(s_t)}\right)^{\xi}\\ -\phi\left[\frac{W_j^i(s_t)}{\pi W_j^i(s_{t-1})}\right]\frac{W_j^i(s_t)}{\pi W_j^i(s_{t-1})}\end{array}\right]$$

$$+\beta E_t u_{c^i}(t+1)\,\Psi^i(t+1)\left[\phi\left[\frac{W_j^i(s_{t+1})}{\pi W_j^i(s_t)}\right]\frac{W_j^i(s_{t+1})}{\pi W_j^i(s_t)}\right],\ \forall j \tag{27}$$

and

$$R(t) = (r(t) - r_d(t)) + \frac{\gamma}{2j^{*i}(t)\,c^i(t)}, \tag{28}$$

where

$$\Psi^i(t) = \left[1 + j^{*i}(t)^2\, R(t) + \left(1 - j^{*i}(t)^2\right)\left(r(t) - r^d(t)\right)\right]^{-1}. \tag{29}$$

Using (9) and $r(t)\,\pi(t) = 1 + R(t)$, it is easy to show that the first-order condition for the household's choice of $j^{*i}(t)$ is

$$\frac{\left[1 + \Gamma\bar{d}(t)^{\theta} + \frac{\gamma}{2j^{*i}(t)c^i(t)}\right]}{r(t)} = \pi(t),$$

suggesting that the optimal choice for the composition of money balances is chosen such that the costs of use for money and deposits are equated. This concludes the endogenous derivation of j_t^* eluded to in footnote 11 of the main text.

Model Solution

The solution methodology described in this appendix follows Lubik and Schorfheide (2003) and their extension of Sims (2001). After removing all multipliers from the household's first-order conditions and imposing symmetry, the normalized system of equations comprising the dynamic solution are given by

$$u_c(t)\Psi(t) -$$

$$\beta E(t)\frac{P(t)}{P(t+1)\mu(t+1)}u_c(t)\left(1+\frac{\gamma}{2j^*(t+1)c(t+1)}+\Gamma d(t+1)^{\theta}\right)\Psi(t+1)=0$$

$$u_c(t)\Psi(t)-\beta E(t)r(t+1)u_c(t+1)\Psi(t+1)=0$$

$$u_h(t)\xi h(t)+u_c(t)\Psi(t)\left[(1-\xi)\frac{W(t)h(t)}{P(t)}-\phi\left(\frac{\mu(t)W(t)}{\pi W(t-1)}-1\right)\frac{\mu(t)W(t)}{\pi W(t-1)}\right]-...$$

$$\beta E(t)\Psi(t+1)\phi\left(\frac{\mu(t+1)W(t+1)}{\pi W(t)}-1\right)\frac{\mu(t+1)W(t+1)}{\pi W(t)}=0$$

$$z(t)=\kappa_z+\rho_z z(t-1)+\varepsilon_z(t)$$

$$\mu(t)=\kappa_\mu+\rho_\mu\mu(t-1)+\varepsilon_\mu(t)$$

$$z(t)k^{\alpha}(t)h^{1-\alpha}(t)+(1-\delta)k(t)=$$

$$c(t)+k(t+1)+\phi\left(\frac{\mu(t)W(t)}{\pi W(t-1)}-1\right)^2+\Gamma d(t)^{1+\theta}+\gamma(1-j^*(t))$$

$$\frac{1}{P(t)}=j^*(t)^2c(t)$$

$$d(t)=\left(1-j^*(t)^2\right)c(t)$$

$$r(t)=\alpha z(t)\left(\frac{h(t)}{k(t)}\right)^{1-\alpha}+1-\delta$$

$$\frac{W(t)}{P(t)}=(1-\alpha)z(t)\left(\frac{k(t)}{h(t)}\right)^{\alpha}$$

where $\Psi(t)=\left[1+\frac{\gamma j^*(t)}{2c(t)}+\Gamma d(t)^{\theta}\right]^{-1}$. After the above system is log-linearized around the model's steady state, the dimension of the system is reduced by using the bottom five equations to remove $\{c(t),h(t),j^*(t),r(t),d(t)\}$. The remaining five equations (and six identities) comprise the linear rational expectations model and can be represented in the canonical form:

$$\Xi_0 s(t) = \Xi_1 s(t-1) + \Upsilon \varepsilon(t) + \Pi \vartheta(t) \tag{30}$$

where

$$\begin{aligned} s(t) &= [k(t+1), W(t), P(t), z(t), \mu(t), E(t)k(t+2), E(t)W(t+1), E(t)P(t+1)]' \\ \varepsilon(t) &= [\varepsilon_z(t), \varepsilon_\mu(t)]' \\ \vartheta(t) &= [k(t+1) - E(t-1)k(t+1), W(t) - E(t-1)W(t), P(t) - E(t-1)P(t)]' \end{aligned}$$

Solving the model requires the use of the generalized Schur decomposition (QZ) of Ξ_0 and Ξ_1. This results in matrices Q, Z, Λ and Ω such that $QQ' = ZZ' = I_n$, Λ and Ω are upper triangular, and $\Xi_0 = Q'\Lambda Z$ and $\Xi_1 = Q'\Omega Z$. Defining $\varpi_t = Z's(t)$, premultiplying (30) by Q results in

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ 0 & \Lambda_{22} \end{bmatrix} \begin{bmatrix} \varpi_{1t} \\ \varpi_{1t} \end{bmatrix} = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ 0 & \Omega_{22} \end{bmatrix} \begin{bmatrix} \varpi_{1t-1} \\ \varpi_{1t-1} \end{bmatrix} + \begin{bmatrix} Q_{1\cdot} \\ Q_{2\cdot} \end{bmatrix} (\Upsilon \varepsilon(t) + \Pi \vartheta(t))$$

where, without loss of generality, the system has been partitioned such that the lower blocks of Λ, Ω and Q correspond to the portion of the system delivering unstable eigenvalues. In other words, the lower block contains all equations in which the ratio between the diagonal elements of Ω and Λ are greater than unity.

This 'explosive' block is written as

$$\varpi_2(t) = \Lambda_{22}^{-1}\Omega_{22}\varpi_{2t-1} + \Lambda_{22}^{-1}Q_{2\cdot}(\Upsilon \varepsilon(t) + \Pi \vartheta(t)).$$

A non-explosive solution of the model requires $\varpi_2(t) = 0 \forall t \geq 0$. This is accomplished by choosing $\varpi_2(0) = 0$ and for every vector $\varepsilon(t)$ the endogenous forecast error $\vartheta(t)$ that satisfies

$$\Upsilon^* \varepsilon(t) + \Pi^* \vartheta(t) = 0 \tag{31}$$

where $\Upsilon^* = Q_{2\cdot}\Upsilon$ and $\Pi^* = Q_{2\cdot}\Pi$. If the number of endogenous forecast errors is equal to the number of unstable eigenvalues, then (31) uniquely determines $\vartheta(t)$. If the number of endogenous forecast errors exceeds the number of unstable eigenvalues, then the system is undetermined and sunspot fluctuations can arise.

Using the singular value decomposition $\Pi^* = UDV'$, a general solution for the endogenous forecast errors is given by

$$\vartheta(t) = \left(-V_{\cdot 1} D_{11}^{-1} U_{\cdot 1}' \Upsilon^* + V_{\cdot 2} M_1\right) \varepsilon(t) + V_{\cdot 2} M_2 \zeta(t)$$

where M_1 and M_2 govern the influence of the sunspot shock.

Assuming Ξ_0^{-1} exists, the solution of the model takes the form of a law of motion for the endogenous variables

(32)

$$s(t) = \Xi_0^{-1}\Xi_1 s(t-1) + \left[\Xi_0^{-1}\Upsilon^* - \Xi_0^{-1}\Pi^* V_{\cdot 1} D_{11}^{-1} U_{\cdot 1}' \Upsilon^*\right]\varepsilon(t) + \Xi_0^{-1}\Pi^* V_{\cdot 2}\left(M_1 \varepsilon(t) + M_2 \zeta(t)\right)\right] .$$

Setting $M_2 = 1$ results in the interpretation of $\zeta(t)$ as a reduced-form sunspot shock. Determining the value for M_1 requires choosing one of two alternative identification schemes. If one assumes that the effects of fundamental and non-fundamental shocks on the forecast error are *orthogonal* to each other, then $M_1 = 0$. Otherwise, M_1 is chosen such that the impulse responses of the model $(\partial s(t) / \partial \varepsilon(t))$ are *continuous* at the boundary between the determinacy and indeterminacy regions. Under indeterminacy, the impulse response is given by

$$B_1 + B_2 M_1 = \left(\Xi_0^{-1}\Upsilon^* - \Xi_0^{-1}\Pi^* V_{\cdot 1} D_{11}^{-1} U_{\cdot 1}' \Upsilon^*\right) + \Xi_0^{-1}\Pi^* V_{\cdot 2} M_1.$$

For a corresponding determinacy solution, the impulse response is given by

$$\tilde{B}_1 = \tilde{\Xi}_0^{-1}\tilde{\Upsilon}^* - \tilde{\Xi}_0^{-1}\tilde{\Pi}^*\tilde{V}_{.1}\tilde{D}_{11}^{-1}\tilde{U}_{.1}'\tilde{\Upsilon}^*$$

where a tilde denotes that a different point in the parameter space is needed to alter the model dynamics. To get the indeterminate impulse responses as close as possible to the determinate ones, M_1 is computed by applying the least squares criterion

$$M_1 = \left[B_2'B_2\right]^{-1} B_2' \left[\tilde{B}_1 - B_1\right].$$

This result is substituted in (32) while maintaining $M_2 = 1$.

Calibration Exercise

Let Φ denote a vector of standard deviations calculated from data, and $\Phi(\Theta)$ denote the corresponding calculations from a simulation of the model where Θ denotes the vector of parameters to be calibrated. The parameter vector delivered by the calibration exercise is that which minimizes

$$\left(\Phi(\Theta) - \Phi\right)' \Sigma \left(\Phi(\Theta) - \Phi\right),$$

where Σ is an identity matrix.

The calibration exercise chooses Φ to be a 3×1 vector consisting of the pre-1984:1 standard deviations of real output, the monetary base and M1 (the data), and Θ is a 3×1 vector of standard deviations of the exogenous shocks of the model (the parameters). Note that minimizing the above expression would be equivalent to a simulated method of moments exercise if Σ were replaced by a weighting matrix that corresponds to the inverse of the variance-covariance matrix of Φ.